Math 2450: Partial Derivatives

What is partial differentiation? Recall that when working with functions of one variable, $f(x)$, the derivative, $f'(x)$, represents the instantaneous rate of change of the function as x changes. When working with functions of two variables, $f(x, y)$, it can be beneficial to examine how the function changes with respect to one of its variables. We need to use **partial differentiation**, a process of differentiating with respect to one variable while keeping the other variable(s) fixed (or constant).

Notation for Partial Derivatives

For $z = f(x, y)$, the partial derivatives f_x and f_y are denoted by

$$f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} f(x, y) = z_x = D_x(f)$$

and

$$f_y(x, y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} f(x, y) = z_y = D_y(f)$$

The values of the partial derivatives of $f(x, y)$ at the point (a, b) are denoted by

$$\frac{\partial f}{\partial x}\Big|_{(a,b)} = f_x(a, b) \quad \text{and} \quad \frac{\partial f}{\partial y}\Big|_{(a,b)} = f_y(a, b)$$

Why is partial differentiation important? Partial differentiation allows us to calculate the slope of a tangent line at a specific point (geometric interpretation) and interpret the value as a rate of change. For example, consider temperature readings at different altitudes up a mountain and times after noon. It is important to note how temperature changes with respect to space (altitude) or time (time after noon). Therefore, we would need to use partial differentiation to find those rates of change.

Example 1. Let $f(x, y) = x^4y^4 + x^2y^3 - xy^2 - y$; determine: **a.** f_x and **b.** f_y

Solution

a. For f_x, we need to hold y constant and find the derivative with respect to x. To help us remember to treat y as a constant, let's let $y = b$:

$$f(x, b) = x^4b^4 + x^2b^3 - xb^2 - b$$

Then by applying the power rule, we take the derivative of each term with respect to x to get:

$$f_x = 4x^3b^4 + 2xb^3 - b^2$$

Substituting y back in for b we get the partial derivative:

$$f_x = 4x^3y^4 + 2xy^3 - y^2$$

Note: The substitution of $y = b$ at the beginning of the solution is not necessary. This was done to help us see that y is a constant when finding the partial derivative with respect to x and to treat it as such. For part b, below, I skip this step.

b. For f_y, we need to hold x constant and find the derivative with respect to y. By applying the power rule, we take the derivative of each term with respect to y to get:

$$f_y = x^4(4y^3) + x^2(3y^2) - x(2y) - 1 = 4x^4y^3 + 3x^2y^2 - 2xy - 1$$

$\star\star$ **Rules for differentiation of one variable functions, $f(x)$, hold for differentiating two variable functions with respect to one of its variables, $f(x, y)$.** These include sum and difference rules, power rule (used above), product rule, quotient rule, and chain rule. Carefully apply product/quotient/chain rule based only on the variable you are considering (as shown in Example 2).

Example 2. Let $f(x, y) = ysin(x^2)$; determine: f_x

Solution

For f_x, we need to hold y constant and find the derivative with respect to x. Since y is a constant, we can use the constant multiple rule of derivatives and chain rule:

$f_x = \frac{\partial}{\partial x}[ysin(x^2)]$	
$f_x = y \cdot \frac{\partial}{\partial x}[sin(x^2)]$	Apply constant multiple rule
$f_x = y \cdot [cos(x^2) \cdot \frac{\partial}{\partial x}(x^2)]$	Apply chain rule with respect to x to find $\frac{\partial}{\partial x}[sin(x^2)]$
$f_x = y \cdot [cos(x^2) \cdot 2x]$	
$f_x = ycos(x^2)2x$	

Example 3. Determine $\frac{\partial z}{\partial y}$ of the function below by differentiating implicitly.

$$x^2 + 2xz + y^2z^2 - z^3 = 4$$

Solution

Here finding the partial derivatives of z is not as simple as it was in example 1. In the function above, we are not given it in the form of $f(x, y) = z$, instead z is being multiplied, raised to powers, and added with the variables x, y. We need to use implicit differentiation (a strategy discussed in Calculus I).

$x^2 + 2xz + y^2z^2 - z^3 = 4$	Given equation
$0 + 2x\frac{\partial z}{\partial y} + 2yz^2 + 2y^2z\frac{\partial z}{\partial y} - 3z^2\frac{\partial z}{\partial y} = 0$	Take derivative of each term with respect to y $x^2 \to 0$ since x is held constant $2xz \to 2x\frac{\partial z}{\partial y}$ by chain rule $y^2z^2 \to 2yz^2 + 2y^2z\frac{\partial z}{\partial y}$ by product rule and chain rule $-z^3 \to -3z^2\frac{\partial z}{\partial y}$ by chain rule
$2yz^2 + \frac{\partial z}{\partial y}[2x + 2y^2z - 3z^2] = 0$	Combine terms with $\frac{\partial z}{\partial y}$
$\frac{\partial z}{\partial y}[2x + 2y^2z - 3z^2] = -2yz^2$	Isolate $\frac{\partial z}{\partial y}$
$\frac{\partial z}{\partial y} = \frac{-2yz^2}{2x+2y^2z-3z^2}$	Divide both sides by $2x + 2y^2z - 3z^2$

Practice Problems

1. Let $f(x, y) = 2x^2 + 3xy - 7y^3 + 4$. Find f_x and f_y.

   [**Solution:** $f_x = 4x + 3y$ and $f_y = 3x - 21y^2$]

2. Let $f(x, y) = sin(x^2) + 3xy$. Find f_x and f_y.

   [**Solution:** $f_x = 2xcos(x^2) + 3y$ and $f_y = 3x$]

3. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ of the function:

$$3x^3 + x^2z + xyz^2 - y^2z^4 = 7$$

   [**Solution:** $\frac{\partial z}{\partial x} = \frac{-9x^2-2xz-yz^2}{x^2+2xyz-4y^2z^3}$ and $\frac{\partial z}{\partial y} = \frac{2yz^4-xz^2}{x^2+2xyz-4y^2z^3}$]